

ANALYSIS OF THE COMPOSITION OF TWO PLANE ISOMETRIES: TRANSLATION, REFLECTION, AND ROTATION USING A LINEAR ALGEBRA APPROACH

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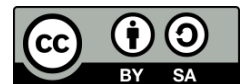
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ABSTRACT

Geometric transformation is a branch of mathematics that focuses on the transformation (bijective function) of geometric objects in a plane or transformation in R^2 . One of the topics discussed in geometric transformation is the properties of the composition of two isometries. Generally, topics in geometric transformation are studied using an axiomatic geometry approach that requires strong geometric visualization skills. This research aims to study the composition of two isometries through an alternative approach, specifically linear algebra. The study focuses on deriving the properties of the composition of two isometries, namely translations, reflections, and rotations, while considering their algebraic forms. The research methodology employed is literature study and deductive reasoning in accordance with mathematical syllogism. The results of this research are theorems that state the properties of the composition of two isometries. Based on the findings of this study, it can be concluded that the properties of the composition of two isometries can be derived using a linear algebra approach.

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1. INTRODUCTION

Transformation geometry is a compulsory course in the Mathematics Education Study Program. It is a branch of mathematics, specifically geometry, that studies transformations [1]. In this research, the focus is on transformations in two dimensions or R^2 . Transformations in R^2 are defined as bijective functions from R^2 to R^2 .

One of the key topics in transformation geometry is isometry, which refers to transformations that preserve distance. Types of isometries studied include translations, reflections, rotations, and glide reflections. An important aspect of isometry is the composition (product) of two isometries, which also preserves distance.

Generally, transformation geometry is taught through an axiomatic geometric approach [2]. This approach involves proving properties of transformation geometry based on axioms in Euclidean geometry, such as the axioms related to line and angle relationships [2]. Teaching through the axiomatic geometric approach requires students to have strong geometric visualization skills. However, in practice, students face difficulties in learning transformation geometry due to challenges in visualizing the concepts involved [3][4].

On the other hand, if transformations are expressed in the form of bijective functions from R^2 to R^2 , some transformations can be represented by the equation $f(x) = Ax + \nu$, where A is an invertible real matrix of order 2×2 and $\nu \in R^2$ for each $x \in R^2$ [5]. Specifically, if $|\det A| = 1$, then f represents an isometry [6]. Based on this fact, the properties of transformation geometry, particularly isometries, can be analyzed using a linear algebra approach. This means that transformation geometry can be studied by applying properties from linear algebra, reducing the need for students to have advanced geometric visualization skills. The properties of the composition of two isometries can be understood algebraically.

In this study, an analysis of the composition of two isometries specifically translations, reflections, and rotations was conducted using a linear algebra approach. The composition of two isometries was analyzed by examining the algebraic form of the isometries, revealing that their composition could be interpreted more clearly and, in certain cases, more easily.

2. RESEARCH METHOD

2.1 Method of gathering data

This study aimed to determine the characteristics of the composition of two isometries using a linear algebra approach. The research method employed was a literature review, with conclusions drawn deductively based on the principles of mathematical syllogism. The results were presented as theorems related to the characteristics of the product of two isometries. These theorems were derived by identifying the simplest algebraic form of the product of two isometries. The research began by gathering relevant theories on isometry equations and linear algebra, particularly matrix properties, bases for R^2 , directional vectors of a line, and relationships between two vectors in R^2 . After obtaining these theories, the next step was to find the simplified form of the product of two isometries. This simplified form was obtained by multiplying the isometric function equations through algebraic manipulation. The algebraic manipulation process utilized properties from linear algebra. Once the simplest algebraic form of the product of two isometries was obtained, the corresponding matrix transformation equations were identified, which led to the determination of the characteristics of the product for each type of isometry composition.

3. RESULT AND ANALYSIS

3.1 Notations

The explanation of the product of two isometries in this paper was obtained by identifying the algebraic forms of the individual isometries. To facilitate the identification process, algebraic manipulation was necessary. Some algebraic expressions are too lengthy to describe in words, which affects the clarity and practicality of writing the article. Therefore, to simplify the writing process and make the research findings more concise, the following notations were defined.

$S_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$: Translation isometry with translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$,

M_t : Reflection isometry with reflection axis along the line t ,

$R_{\alpha, \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$: Rotation isometry with rotation angle of α and center of rotation $\begin{pmatrix} a \\ b \end{pmatrix}$,

$G_{t, \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$: Glide reflection isometry formed by M_t and $S_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}$.

$$[r_\alpha] = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

$$[m_\alpha] = \begin{pmatrix} -\cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix},$$

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

3.2 Background Theory

3.2.1 Geometry Transformation

A transformation is a bijective function from a set to the same set [7]. In this article, the set in question is R^2 . A transformation on R^2 is referred to as a geometric transformation.

Definition 1. T is a transformation if and only if $T : R^2 \rightarrow R^2$ is bijective [8]. One concept of geometric transformation discussed in this article is the invariant point. An invariant point is a point that is mapped to itself by a transformation.

Definition 2. $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ is an invariant point of the transformation T if $T \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ [9].

3.2.2 Isometry

Isometry is a type of geometric transformation. A geometric transformation is called an isometry if it preserves distances [10].

Definition 3. A geometric transformation T is called an isometry if $\forall x, y \in R^2, d(x, y) = d(T(x), T(y))$ [11].

There are three types of isometries: $S_{\begin{pmatrix} a \\ b \end{pmatrix}}$, $R_{\alpha, \begin{pmatrix} a \\ b \end{pmatrix}}$, and $G_{t, \begin{pmatrix} a \\ b \end{pmatrix}} = M_t S_{\begin{pmatrix} a \\ b \end{pmatrix}}$ [11].

3.2.3 Algebraic Forms of Isometries

As geometric transformations, isometries have algebraic forms as functions. Each type of isometry has a distinct algebraic form.

Definition 4. The statements $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; S_{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}; \begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ represent a translation transformation.

Definition 5. The statements $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; M_t \begin{pmatrix} x \\ y \end{pmatrix} = [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}; t : x \cos \alpha + y \sin \alpha - p = 0; \alpha, p \in R$ represent a reflection transformation across line t .

Definition 6. The statement $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; R_{\theta, \begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = [r_\theta] \begin{pmatrix} x - a \\ y - b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}; \theta \in R$ and $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ represent a rotation transformation centered at $\begin{pmatrix} a \\ b \end{pmatrix}$ with a rotation angle of θ .

Definition 7. The statements $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; G_{t, \begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = M_t S_{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = S_{\begin{pmatrix} a \\ b \end{pmatrix}} M_t \begin{pmatrix} x \\ y \end{pmatrix}; \begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ and $\begin{pmatrix} a \\ b \end{pmatrix} \parallel t$ represent a glide reflection transformation [3].

3.2.4 Composition of two isometries

The composition of two isometries refers to the combination of two isometries, which is itself a function. If T_1 and T_2 are isometries, then the composition of T_1 and T_2 , denoted by $T_2 T_1$, is defined as follows:

Definition 8. $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; T_1 T_2 \begin{pmatrix} x \\ y \end{pmatrix} = T_1 \left(T_2 \begin{pmatrix} x \\ y \end{pmatrix} \right)$ [2].

3.2.5 Basis for R^2

Theorem 3.2.5.1. For every $\alpha \in [0, 2\pi)$, $\left\{ \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \right\}$ is a basis for R^2 .

Proof. Let $\alpha \in [0, 2\pi)$, and $\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$ be the column vectors of $[r_\alpha]$. Notice that $\det[r_\alpha] = 1$, thus

$\left\{ \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \right\}$ forms a basis for R^2 [4].

If $\left\{ \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \right\}$ is a basis for R^2 , then $\left\{ \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \right\}$ spans R^2 . Therefore, $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \exists m, n \in$

R such that $\begin{pmatrix} x \\ y \end{pmatrix} = m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$ [4].

3.2.6 Vectors and Straight Lines in R^2

In this paper, the understanding of vectors and straight lines in R^2 is essential because it relates to the algebraic forms of certain types of isometries. Additionally, the understanding of vectors and lines in R^2 is used in the algebraic manipulation process to identify the characteristics of the composition of two isometries.

Definition 9. Every straight line in two dimensions can be expressed in the Hesse Normal Form $x \cos \alpha + y \sin \alpha - p = 0; \alpha, p \in R$ [8].

A line $s : x \cos \alpha + y \sin \alpha - p = 0; \alpha, p \in R$ has a direction vector $\begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$ and a normal vector $\begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ [9].

Therefore, a vector $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ is said to be parallel to s if $m \in R$ and then $\begin{pmatrix} a \\ b \end{pmatrix} = m \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$.

Furthermore, a vector $\begin{pmatrix} c \\ d \end{pmatrix} \in R^2$ is said to be perpendicular to s if there exists $n \in R$ such that $\begin{pmatrix} c \\ d \end{pmatrix} = n \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$.

3.3 Characteristics of the Composition of Two Isometries

This section explains the properties of the composition of two isometries, specifically translations, reflections, and rotations, by considering the algebraic equations of each. In this article, the properties of the composition of two isometries (translations, reflections, and rotations) are expressed in the form of theorems.

3.3.1 Composition of Two Translations

Theorem 3.3.1.1. The composition of two translations is a translation with a translation vector in the same direction as the sum of the two translation vectors.

Proof. Let $S_{\begin{pmatrix} a \\ b \end{pmatrix}}$ and $S_{\begin{pmatrix} c \\ d \end{pmatrix}}$ be two translations with $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in R^2$ as their respective translation directions.

When these two translations are composed, the result is as follows:

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, S_{\begin{pmatrix} a \\ b \end{pmatrix}} S_{\begin{pmatrix} c \\ d \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} = S_{\begin{pmatrix} a \\ b \end{pmatrix}} \left(S_{\begin{pmatrix} c \\ d \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} \right) = S_{\begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x+c \\ y+d \end{pmatrix} = \begin{pmatrix} x+c+a \\ y+d+b \end{pmatrix} = S_{\begin{pmatrix} a+c \\ b+d \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}.$$

3.3.2 Composition of Two Reflections

Theorem 3.3.2.1. If two reflections have parallel axes, their composition results in a translation. The translation produced is in the direction of the initial translation vector, with a magnitude equal to half the distance between the two reflection axes, either in the same or opposite direction as the original vector.

Proof. Let M_s and M_t be the two reflections, with parallel reflection axes $s : x \cos \beta + y \sin \beta - p = 0, t : x \cos \beta + y \sin \beta - q = 0$, and $p, q, \beta \in R$.

$$\begin{aligned}
 M_s M_t \begin{pmatrix} x \\ y \end{pmatrix} &= M_s \left([m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + 2q \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \right) \\
 &= [m_{2\beta}] \left([m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + 2q \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \\
 &= [m_{2\beta}]^2 \begin{pmatrix} x \\ y \end{pmatrix} + 2q[m_{2\beta}] \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} + 2p \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \\
 &= \begin{pmatrix} x \\ y \end{pmatrix} - 2q \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} + 2p \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \\
 &= \begin{pmatrix} x \\ y \end{pmatrix} + 2(p - q) \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \\
 &= S_{2(p-q)} \begin{pmatrix} x \\ y \end{pmatrix}.
 \end{aligned}$$

Theorem 3.3.2.2. The composition of two reflections produces a rotation if their reflection axes are not parallel. The angle of rotation is twice the angle formed by the two reflection axes, and the center of rotation is the intersection point of the axes.

Proof. Let M_s and M_t be the two reflections with axes $s : x \cos \alpha + y \sin \alpha - p = 0$, $t : x \cos \beta + y \sin \beta - q = 0$; $p, q, \alpha, \beta \in R$. The intersection point of the two reflection axes is $\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \cos \beta & \sin \beta \end{pmatrix}^{-1} \begin{pmatrix} p \\ q \end{pmatrix}$

$$\begin{aligned}
 \forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2 \exists R_{2(\alpha-\beta)} \begin{pmatrix} x \\ y \end{pmatrix} &= [r_{2(\alpha-\beta)}] \begin{pmatrix} x - k \\ y - l \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} \\
 &= [r_{2(\alpha-\beta)}] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_{2(\alpha-\beta)}]) \begin{pmatrix} k \\ l \end{pmatrix} \\
 &= [m_{2\alpha}][m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + \left(I_2 - [r_{2(\alpha-\beta)}] \begin{pmatrix} \cos \alpha & \sin \alpha \\ \cos \beta & \sin \beta \end{pmatrix}^{-1} \right) \begin{pmatrix} p \\ q \end{pmatrix} \\
 &= [m_{2\alpha}][m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \cos \alpha - \cos(2\alpha - \beta) \\ \sin \alpha - \sin(2\alpha - \beta) \end{pmatrix} \begin{pmatrix} 2p \\ 2q \end{pmatrix} \\
 &= [m_{2\alpha}][m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + 2q \begin{pmatrix} -\cos(2\alpha - \beta) \\ -\sin(2\alpha - \beta) \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
 &= [m_{2\alpha}] \left([m_{2\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + 2q \begin{pmatrix} \cos \beta \\ \sin \beta \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
 &= [m_{2\alpha}] \left(m_t \begin{pmatrix} x \\ y \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
 &= M_s M_t \begin{pmatrix} x \\ y \end{pmatrix}.
 \end{aligned}$$

3.3.3 Composition of Two Rotations

Theorem 3.3.3.1. The composition of two rotations will produce a translation if the sum of their rotation angles equals one full rotation.

Proof. Let $R_{\alpha.(p/q)}$ and $R_{\beta.(u/v)}$ be the two rotations, with $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$ as the center of rotation and α, β as the rotation angles such that $\alpha + \beta = 2\pi$.

$$\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2; \text{ if } \alpha + \beta = 2\pi \text{ with } \alpha, \beta \in R \text{ and } n \in Z, \text{ then } R_{\alpha.(p/q)} R_{\beta.(u/v)} \begin{pmatrix} x \\ y \end{pmatrix} = S_{(k)} \begin{pmatrix} x \\ y \end{pmatrix} \text{ with}$$

$$\begin{pmatrix} k \\ l \end{pmatrix} = ([r_\alpha] - I_2) \begin{pmatrix} u - p \\ v - q \end{pmatrix}; \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$$

Proof. Let $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$ be the centers of rotation and $\alpha, \beta \in R$ represent the rotation angles with $\alpha + \beta = 2\pi$. Then, the following equation holds:

$$\begin{aligned} \forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2 \exists R_{\alpha, \begin{pmatrix} p \\ q \end{pmatrix}} R_{\beta, \begin{pmatrix} u \\ v \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} &= R_{\alpha, \begin{pmatrix} p \\ q \end{pmatrix}} \left([r_\beta] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \right) \\ &= [r_\alpha] \left([r_\beta] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right) + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= [r_\alpha][r_\beta] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + [r_\alpha] \begin{pmatrix} u - p \\ v - q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= [r_{\alpha+\beta}] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + [r_\alpha] \begin{pmatrix} u - p \\ v - q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} + [r_\alpha] \begin{pmatrix} u - p \\ v - q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} + [r_\alpha] \begin{pmatrix} u - p \\ v - q \end{pmatrix} - I_2 \begin{pmatrix} u - p \\ v - q \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} + ([r_\alpha] - I_2) \begin{pmatrix} u - p \\ v - q \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix} = ([r_\alpha] - I_2) \begin{pmatrix} u - p \\ v - q \end{pmatrix} \\ &= S_{\begin{pmatrix} k \\ l \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} k \\ l \end{pmatrix} = ([r_\alpha] - I_2) \begin{pmatrix} u - p \\ v - q \end{pmatrix}. \end{aligned}$$

Theorem 3.3.3.2. The composition of two rotations will produce a rotation if the sum of their rotation angles does not equal one full rotation.

Proof. Let $\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$ be the centers of rotation and $\alpha, \beta \in R$ represent the magnitude of the rotation angles with $\alpha + \beta \neq 2\pi$. Then, the following statement holds for $\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + ([r_{\alpha+\beta}] - I_2)^{-1} ([r_\alpha] - I_2) \begin{pmatrix} p - u \\ q - v \end{pmatrix}$:

$$\begin{aligned} \forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, R_{(\alpha+\beta), \begin{pmatrix} k \\ l \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} &= [r_{\alpha+\beta}] \begin{pmatrix} x - k \\ y - l \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} \\ &= [r_{\alpha+\beta}] \begin{pmatrix} x \\ y \end{pmatrix} - ([r_{\alpha+\beta}] - I_2) \begin{pmatrix} k \\ l \end{pmatrix} \\ &= [r_{\alpha+\beta}] \begin{pmatrix} x \\ y \end{pmatrix} - ([r_{\alpha+\beta}] - I_2) \begin{pmatrix} u \\ v \end{pmatrix} - ([r_\alpha] - I_2) \begin{pmatrix} p - u \\ q - v \end{pmatrix} \\ &= [r_\alpha][r_\beta] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} - [r_\alpha] \begin{pmatrix} p - u \\ q - v \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} u \\ v \end{pmatrix} \\ &= [r_\alpha] \left([r_\beta] \begin{pmatrix} x - u \\ y - v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right) + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= [r_\alpha] \left(R_{\beta, \begin{pmatrix} u \\ v \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} p \\ q \end{pmatrix} \right) + \begin{pmatrix} p \\ q \end{pmatrix} \\ &= R_{\alpha, \begin{pmatrix} p \\ q \end{pmatrix}} R_{\beta, \begin{pmatrix} u \\ v \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

3.3.4 Composition of Translation and Reflection

Theorem 3.3.4.1. The composition of a translation in the direction of the reflection symmetry axis results in a reflection.

Proof. Let $S_{\binom{u}{v}}$ be a translation in the direction of $\binom{u}{v} = m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$, and M_S be a reflection with respect to the symmetry axis: $x \cos \alpha + y \sin \alpha - p = 0$ and $\binom{u}{v} = m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$; $\binom{u}{v} \in R^2$ and $m, p, \alpha \in R$. Then, the following statement holds for

$$\begin{aligned} \forall \binom{x}{y} \in R^2, \exists M_S S_{\binom{u}{v}} \binom{x}{y} &= M_S \binom{x+u}{y+v} \\ &= [m_{2\alpha}] \binom{x+u}{y+v} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= [m_{2\alpha}] \binom{x}{y} + [m_{2\alpha}] \binom{u}{v} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= [m_{2\alpha}] \binom{x}{y} + [m_{2\alpha}] \left(m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= [m_{2\alpha}] \binom{x}{y} - m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= [m_{2\alpha}] \binom{x}{y} + 2 \left(p - \frac{m}{2} \right) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= M_{t_1} \binom{x}{y}; \quad t_1 : x \cos \alpha + y \sin \alpha - \left(p - \frac{m}{2} \right) = 0. \end{aligned}$$

Additionally, the following statement also holds.

$$\begin{aligned} \forall \binom{x}{y} \in R^2, \exists S_{\binom{u}{v}} M_S \binom{x}{y} &= S_{\binom{u}{v}} \left([m_{2\alpha}] \binom{x}{y} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right) \\ &= [m_{2\alpha}] \binom{x}{y} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + \binom{u}{v} \\ &= [m_{2\alpha}] \binom{x}{y} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= [m_{2\alpha}] \binom{x}{y} + 2 \left(p + \frac{m}{2} \right) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\ &= M_{t_2} \binom{x}{y}; \quad t_2 : x \cos \alpha + y \sin \alpha - \left(p + \frac{m}{2} \right) = 0. \end{aligned}$$

Theorem 3.3.4.2. The composition of a translation and a reflection that are not in the direction of the reflection symmetry axis results in a glide reflection or translation reflection.

Proof. Let $S_{\binom{u}{v}}$ be a translation in the direction of $\binom{u}{v}$, and M_S be a reflection with respect to the symmetry axis: $x \cos \alpha + y \sin \alpha - p = 0$ and $\binom{u}{v} = m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}$, $\binom{u}{v} \in R^2$, and $m, n, p, \alpha \in R$, $n \neq 0$, then the following statement holds:

$$\begin{aligned}
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \exists M_S S_{\begin{pmatrix} u \\ v \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} &= M_S \begin{pmatrix} x+u \\ y+v \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x+u \\ y+v \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + [m_{2\alpha}] \begin{pmatrix} u \\ v \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + [m_{2\alpha}] \left(m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} - m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2 \left(p - \frac{m}{2} \right) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \\
&= M_{t_1} \begin{pmatrix} x \\ y \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}, \quad t_1 : x \cos \alpha + y \sin \alpha - \left(p - \frac{m}{2} \right) = 0 \\
&= S_{n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}} M_{t_1} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t_1 : x \cos \alpha + y \sin \alpha - \left(p - \frac{m}{2} \right) = 0 \\
&= G_{t_1, n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}, \quad t_1 : x \cos \alpha + y \sin \alpha - \left(p - \frac{m}{2} \right) = 0.
\end{aligned}$$

Furthermore, the following statement also holds.

$$\begin{aligned}
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \exists S_{\begin{pmatrix} u \\ v \end{pmatrix}} M_S \begin{pmatrix} x \\ y \end{pmatrix} &= S_{\begin{pmatrix} u \\ v \end{pmatrix}} \left([m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right) \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + m \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \\
&= [m_{2\alpha}] \begin{pmatrix} x \\ y \end{pmatrix} + 2 \left(p + \frac{m}{2} \right) \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix} \\
&= M_{t_2} \begin{pmatrix} x \\ y \end{pmatrix} + n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}; \quad t_2 : x \cos \alpha + y \sin \alpha - \left(p + \frac{m}{2} \right) = 0 \\
&= S_{n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}} M_{t_2} \begin{pmatrix} x \\ y \end{pmatrix}; \quad t_2 : x \cos \alpha + y \sin \alpha - \left(p + \frac{m}{2} \right) = 0 \\
&= G_{t_2, n \begin{pmatrix} -\sin \alpha \\ \cos \alpha \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}; \quad t_2 : x \cos \alpha + y \sin \alpha - \left(p + \frac{m}{2} \right) = 0.
\end{aligned}$$

3.3.5 Composition of Translation and Rotation

Theorem 3.3.5. The composition of a translation and a rotation results in a rotation.

Proof. Let $S_{\begin{pmatrix} u \\ v \end{pmatrix}}$ be a translation in the direction of $\begin{pmatrix} u \\ v \end{pmatrix}$, and $R_{\alpha, \begin{pmatrix} a \\ b \end{pmatrix}}$ be a rotation centered at $\begin{pmatrix} a \\ b \end{pmatrix}$ with an angle of rotation equal to α with $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$ and $\alpha \in R$. Let it be established that the value of $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + (I_2 - [r_\alpha])^{-1} [r_\alpha] \begin{pmatrix} u \\ v \end{pmatrix}$ holds. This conditioning results in the following statement holding:

$$\begin{aligned}
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \exists R_{\alpha, (p)} \begin{pmatrix} x \\ y \end{pmatrix} &= [r_\alpha] \begin{pmatrix} x-p \\ y-q \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \begin{pmatrix} p \\ q \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \left(\begin{pmatrix} a \\ b \end{pmatrix} + (I_2 - [r_\alpha])^{-1} [r_\alpha] \begin{pmatrix} u \\ v \end{pmatrix} \right) \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \begin{pmatrix} a \\ b \end{pmatrix} + [r_\alpha] \begin{pmatrix} u \\ v \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x-a+u \\ y-b+v \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \\
&= [r_\alpha] \left(S_{(v)} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \right) + \begin{pmatrix} a \\ b \end{pmatrix} \\
&= R_{\alpha, (a)} S_{(v)} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

Next, for $\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + (I_2 - [r_\alpha])^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$, the following statement holds.

$$\begin{aligned}
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2, \exists R_{\alpha, (l)} \begin{pmatrix} x \\ y \end{pmatrix} &= [r_\alpha] \begin{pmatrix} x-k \\ y-l \end{pmatrix} + \begin{pmatrix} k \\ l \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \begin{pmatrix} k \\ l \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \left(\begin{pmatrix} a \\ b \end{pmatrix} + (I_2 - [r_\alpha])^{-1} \begin{pmatrix} u \\ v \end{pmatrix} \right) \\
&= [r_\alpha] \begin{pmatrix} x \\ y \end{pmatrix} + (I_2 - [r_\alpha]) \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \\
&= [r_\alpha] \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \\
&= R_{\alpha, (a)} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \\
&= S_{(v)} R_{\alpha, (a)} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

3.3.6 Composition of Rotation Followed by Reflection

Theorem 3.3.6. The composition of a reflection and a rotation results in a glide reflection.

Proof. Let M_S be a reflection with respect to the symmetry axis $s : x \cos \alpha + y \sin \alpha - p = 0$, and $R_{\beta, (a)}$ be a rotation centered at $\begin{pmatrix} a \\ b \end{pmatrix}$ with an angle of rotation equal to β with $\begin{pmatrix} a \\ b \end{pmatrix} \in R^2$ and $\beta \in R$, then the following statement holds:

$$\begin{aligned}
\forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \exists M_S R_{\beta, \begin{pmatrix} a \\ b \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} &= M_S \left([r_\beta] \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) \\
&= [m_{2\alpha}] \left([r_\beta] \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \right) + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha}] [r_\beta] \begin{pmatrix} x \\ y \end{pmatrix} + [m_{2\alpha}] (I_2 - [r_\beta]) \begin{pmatrix} a \\ b \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \\
&= [m_{2\alpha-\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + [r_{\alpha-\frac{\beta}{2}}] [r_{\alpha-\frac{\beta}{2}}]^{-1} \left([m_{2\alpha}] (I_2 - [r_\beta]) \begin{pmatrix} a \\ b \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right) \\
&= [m_{2\alpha-\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + [r_{\alpha-\frac{\beta}{2}}]^{-1} \begin{pmatrix} k \\ l \end{pmatrix}; \\
\begin{pmatrix} k \\ l \end{pmatrix} &= [r_{\alpha-\frac{\beta}{2}}]^{-1} \left([m_{2\alpha}] (I_2 - [r_\beta]) \begin{pmatrix} a \\ b \end{pmatrix} + 2p \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} \right) \\
&= [m_{2\alpha-\beta}] \begin{pmatrix} x \\ y \end{pmatrix} + k \begin{pmatrix} -\sin \left(\alpha - \frac{\beta}{2} \right) \\ \cos \left(\alpha - \frac{\beta}{2} \right) \end{pmatrix} + l \begin{pmatrix} -\sin \left(\alpha - \frac{\beta}{2} \right) \\ \cos \left(\alpha - \frac{\beta}{2} \right) \end{pmatrix} \\
&= M_t \begin{pmatrix} x \\ y \end{pmatrix} + l \begin{pmatrix} -\sin \left(\alpha - \frac{\beta}{2} \right) \\ \cos \left(\alpha - \frac{\beta}{2} \right) \end{pmatrix}; t : x \cos \left(\alpha - \frac{\beta}{2} \right) + y \sin \left(\alpha - \frac{\beta}{2} \right) - \frac{k}{2} = 0 \\
&= S_{l \begin{pmatrix} -\sin \left(\alpha - \frac{\beta}{2} \right) \\ \cos \left(\alpha - \frac{\beta}{2} \right) \end{pmatrix}} M_t \begin{pmatrix} x \\ y \end{pmatrix} \\
&= G_{t, l \begin{pmatrix} -\sin \left(\alpha - \frac{\beta}{2} \right) \\ \cos \left(\alpha - \frac{\beta}{2} \right) \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}
\end{aligned}$$

4. CONCLUSION

Based on the discussion of the research results, it can be concluded that geometric transformations, particularly the composition of two isometries—namely translations, reflections, and rotations can be thoroughly discussed using a linear algebra approach. The characteristics of the composition of two isometries can be identified based on the algebraic form of each isometry involved. This indicates that linear algebra can serve as an alternative approach for studying geometric transformations, in addition to the axiomatic geometric approach. The main conclusion of this research is that the analysis of the characteristics of the composition of two isometries, which include translations, reflections, and rotations, can be effectively discussed using algebraic methods. In some cases, proving theorems using an algebraic approach can reduce the difficulty of the proof process, as the discussion is more analytical, easily visualized (for instance, by utilizing applications like GeoGebra), and simplifies the proof process in terms of clarity and logical simplicity.

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